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THE SEQUENCE OF CODIMENSIONS OF PI-ALGEBRAS

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ABSTRACT

Bounds and asymptotic formulas are given for the size of the irreducible representations of the symmetric groups. These are applied to obtain information on the identities and codimension sequence $c_n(R)$ of a PI-algebra R of characteristic zero, e.g., the "ultimate" width of the hook in which the diagrams of the cocharacters of R lies is $\leqq (\underline{\lim}\, c_n(R)^{1/n})^2$, and $\overline{\lim}\, c_n(R)^{1/n} \leqq 2(\underline{\lim}\, c_n(R)^{1/n})^2$ for rings with no right (or left) total annihilators.

1. Introduction

The relation between the representations of the symmetric group and the polynomial identities $I(R)$ of an algebra R of characteristic zero, has been developed by Regev in a sequence of papers. The basic idea is to consider the set of multilinear homogeneous polynomials V_n in n non-commutative indeterminates $x_1, \cdots, x_n$ as an S_n-module, isomorphic with FS_n, by setting $\sigma f[x_1, \cdots, x_n] = f[x_{\sigma(1)}, \cdots, x_{\sigma(n)}]$, the polynomials of V_n which are identities of R, i.e. $I_n(R) = I(R) \cap V_n$ is a left S_n-module, and by identifying V_n with the group ring FS_n, $I_n(R)$ is a left ideal in FS_n. The n-th cocharacter $x_n(R)$ is the character of the quotient module $V_n/I_n(R)$, and its dimension $c_n(R)$ is the n-th codimension of R. $V_n/I_n(R)$ is a direct sum of irreducible left ideals I_D, and $c_n(R) = \Sigma\, a_D \dim D$. In particular, if for some Young diagram D', $\dim D' > c_n(R)$, then the two-sided ideal $I_{D'}$ is included in $I_n(R)$. Regev in [4] has shown that the classical result, that R satisfies a power of standard polynomial $S_h[x_1, \cdots, x_h]^k = 0$, can be shown with $h = (d-1)^2 + 1$, and $k \sim h^4$ where d is the degree of a minimal identity of R. In [1], Regev and the author have shown that the diagrams D of the cocharacter lie in a hook of width $\sim e(d-1)^4$.

The present paper uses the methods developed by Regev and in [1], to obtain more refined results in this direction.

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First we obtain better bounds for $\dim(k^h)$ of a Young diagram of rectangles $k \times h$, and of hooks shown in Fig. 2. These bounds are used to show, e.g., that $S_h^k[x] = 0$ will hold for $k \sim h^2 \log h$ which is better than the bound of [4].

Next we consider $\underline{\lim}\, c_n(R)^{1/n} = c$, and prove that c can replace Latyshef bound $(d-1)^2$ which was used in [1] and [4], but then without giving precise bounds, but rather results of the form, e.g.: (1) for each $h > c$ there exists k such that R satisfies $S_h[x]^k = 0$; (2) the diagrams D of the cocharacter lie in a hook of the shape in Fig. 3 (Corollary 9) with 'ultimate' width $\leqq c^2$. An interesting corollary is that $\underline{\lim}\, c_n(R)^{1/n}$ and $\overline{\lim}\, c_n(R)^{1/n}$ are not independent. For matrix rings and for the exterior algebra actually $\lim c_n(R)^{1/n}$ exists (Regev [5], Drensky [3]). In the general case we could only prove that $\overline{\lim}\, c_n(R)^{1/n} \leqq 2(\underline{\lim}\, c_n(R)^{1/n})^2$. A lower bound for c is s^2, where s is a size of matrices in which $R/N_1(R)$ can be embedded, where $N_1(R)$ is the sum of all nilpotent ideals of R.

2. Dimensions of the representations of S_n

All algebras and representations in this paper are over fields of characteristic zero. Let D be a Young diagram of content n; $\dim D$ will denote the dimension of the corresponding representation of the symmetric group S_n.

To compute a lower bound for $\dim D$ we use the hook formula

$$\dim D = \frac{n!}{\prod h_{ij}} \tag{2.1}$$

where h_{ij} is the hook number, that is, the number of squares in the hook through the (i,j) square. Let s, t denote the number of squares of the corresponding legs (Fig. 1); then $h_{ij} = s + t - 1$. Hence we obtain from (2.1)

$$\begin{aligned} \log\dim D &= \sum_{\nu=1}^{n} \log \nu - \sum \log(s+t-1) \\ &\geqq n((\log n) - 1) - \iint \log^+(s+t-1)\,ds\,dt \end{aligned} \tag{2.2}$$

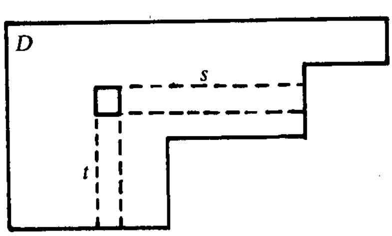

Fig. 1.

where $\log^+ n = \log n$ for $n \geqq 1$ and zero elsewhere, and the integral is taken over the area of D with each square a unit square.

Let $T = (k^h)$ be a rectangle of h rows each of k squares. For T it is easy to obtain an upper bound for the integral:

$$\iint \log^+(s+y-1)dsdt \leqq \int_0^k \int_0^h \log^+(s+t)dsdt$$
$$= \int_0^k [(s+t)(\log^+(s+t)-1)]_0^h ds$$
$$= \tfrac{1}{2}(k+h)^2 \log(k+h) - \tfrac{1}{2}k^2 \log k - \tfrac{1}{2}h^2 \log h - \tfrac{3}{2}kh$$

which is obtained by using the equation $\int x(\log x - 1)dx = \frac{1}{2}x^2(\log x - \frac{3}{2}) + C.$

Let $x = k/h$, and divide (2.2) by $n = kh$; using the inequality $\Sigma_{\nu=1}^n \log \nu \geqq n(\log n - 1)$:

$$\frac{1}{n}\log \dim T \geqq \log xh^2 - 1 - \frac{1}{2x}(1+x)^2 \log(1+x)h + \frac{x}{2}\log xh + \frac{1}{2x}\log h + \frac{3}{2}$$
$$= \log h - \varphi(x) \tag{2.3}$$

where

$$\varphi(x) = \frac{(1+x)^2}{2x}\log(1+x) - \left(1+\frac{x}{2}\right)\cdot \log x - \tfrac{1}{2}$$
$$= \frac{1}{2x}[F(x+1) - F(x) - F'(x)] \tag{2.4}$$
$$= \frac{1}{2x}\int_0^1 (1-t)F''(x+t)dt$$

with $F(x) = x^2 \log x$. Thus

$$\varphi(x) = \frac{1}{2x}\int_0^1 (1-t)(2\log(x+t)+3)dt$$
$$\leqq \frac{1}{2x}(\log(x+1) + \tfrac{3}{2}). \tag{2.5}$$

A simple bound for $\varphi(x)$ is $A(\log x)/x$, where A is a constant whose value can be chosen suitably, if the values of x are not too near to 1. It follows easily by computation that $\varphi(x) \leqq (\log x)/x$ for $x \geqq 4.8$, and, e.g., $\varphi(X) \leqq 2(\log x)/x$ for $x \geqq 1.75$. In fact, for $\varepsilon > 0$ there exists x_0 so that

$$\varphi(x) \leqq (\tfrac{1}{2}+\varepsilon)\frac{\log x}{x} \qquad \text{for } x \geqq x_0,$$

since the integral of (2.5) is $\geqq \log x$.

Summarizing, we obtain:

THEOREM 1. *If $T=(k^h)$, then* $\dim T \geqq (e^{-\varphi(x)}h)^{kh}$, $x=k/h$ *and* $\varphi(x)\to 0$ *as* $x\to\infty$. *Furthermore, for* $x \geqq 4.8$, $\dim T \geqq (x^{-1/x}h)^n$.

An asymptotic formula and bounds for some other diagrams will be given later.

The next lemma is readily proved by a straightforward computation using the inequality $\log(1+x)<x$ for $x>0$, or by using Newton's method to obtain a bound for the solution of the equation $x/M-\log x=0$, $M>0$.

LEMMA 2. *If* $1<M\leqq e$ *and* $x>M\log M$, *or if* $M>e$ *and*

$$x > M\log M\left(1+\frac{\log\log M}{\log M-1}\right)$$

then $(\log x)/x<1/M$.

3. First applications to PI

Let V_n be the linear space of all n-homogeneous multilinear polynomials in n non-commutative indeterminates, and consider it as the S_n-left module (S_n the permutation group) given by $\sigma f[x_n,\cdots,x_n]=f[x_{\sigma(1)},\cdots,x_{\sigma(n)}]$.

Let $I(R)$ be the set of all identities of an algebra R over a field of characteristic zero. The sequence of codimensions of R is defined by Regev: $c_n(R)=\dim(V_n/I_n(R))$, where $I_n(R)=V_n\cap I(R)$.

First we follow Regev's method of [4], but we use our bounds of $\dim D$ to obtain a *more* refined result than those of [4]. We begin by quoting some of the basic facts of [1] and [4]:

Given a Young diagram of content $|D|=n$, let I_D denote the ideal corresponding to D in the group algebra FS_n, and we identify FS_n with V_n by identifying $\Sigma\,\alpha_\sigma\sigma$ with $\Sigma\,\alpha_\sigma x_{\sigma(1)}\cdots x_{\sigma(n)}$. A basic lemma of [4] which is the main tool is:

(3.1) If $c_n(R)<\dim D$ then $I_D\subseteq I_n(R)$.

This has the corollary (Regev [4]):

If $c_n(R)\leqq \dim T$, $T=(k^h)$, then R satisfies the identity $S_h[x_1,\cdots,x_h]^k=0$,

(3.2) where $S_h[x]$ is the standard polynomial.

And by [1]:

If $I(R) \supseteq I_{D'}$ for all $D' \geqq D$, and $|D'| = n + m$
then R satisfies all identities of the form

$$f^*[x] = \sum \alpha_\sigma m_{i_0} x_{\sigma(1)} m_{i_1} x_{\sigma(2)} \cdots m_{i_{n-1}} x_{\sigma(n)} m_{i_n} \tag{3.3}$$

for all $f = \sum \alpha_\sigma \sigma \in I_D$, and all monomials m_j in $1, x_{n+1}, \cdots, x_{n+m}$.

In particular we obtain:

PROPOSITION 3. *If $I(R) \supseteq I_D$ for all Young diagrams D containing the rectangle $T = (k^h)$ where $|D| = k(h+1)$, then the principle ideal generated by any element $S_h[a_1, \cdots, a_h]$, $a_i \in R$ is nilpotent of index $\leqq k$. Hence, if $N(R)$ is the sum of all nilpotent ideals of R then $R/N(R)$ satisfies the identity $S_h[x] = 0$.*

Indeed, it follows from (3.3) and (3.2) that R satisfies the identity (with k factors)

$$S_h[x_1, \cdots, x_h] y_1 S_h[x_1, \cdots, x_h] y_2 \cdots S_h[x_1, \cdots, x_h] y_h = 0,$$

and the rest follows immediately.

We shall also need the following result of [1]:

$$\text{If for all diagrams } D' \geqq D, 2|D| > |D'| \geqq |D| \text{ the identities } I(R) \supseteq I_{D'}, \text{ then for all } D' \geqq D, I(R) \supseteq I_{D'}. \tag{3.4}$$

Our first result is a refinement of a result of Regev [4] who has proved the next theorem for $k \sim h^4$:

THEOREM 4. (i) *Let R be a PI algebra satisfying an identity of degree $d \geqq 3$, then for every $h \geqq 1 + (d-1)^2$ and*

$$k > h^2 \log h \left(1 + \frac{\log\log h}{\log h - 1}\right),$$

the ring R satisfies the identities $S_h^k[x] = 0$ and $S_k[x]^h = 0$.

(ii) *R satisfies also the identity $S_h[x]^h = 0$, for $h = [4e^{-1/2}(d-1)^2] + 1$.*

PROOF. We follow Regev's method of [4] using our bounds, to show that for k, h of our theorem we have $\dim T > c_n(R)$ and apply (3.2).

Indeed, assume first that $x = k/h \geqq 4.9$, Using the bound of Latyshef that $c_n(R) \leqq (d-1)^{2n}$, we have to show, by Theorem 1, that

$$\frac{1}{n} \log \dim T > \log h - \varphi(x) \geqq \log h - \frac{\log x}{x} \geqq \log(d-1)^2$$

or equivalently that

$$\frac{\log x}{x} < \log \frac{h}{(d-1)^2}.$$

If $h \geqq 1+(d-1)^2$, then $\log(h/(d-1)^2) \geqq 1/h$ hence it suffices to show that $(\log x)/x \leqq 1/h$. We can apply Lemma 2, since $h \geqq 1+(d-1)^2 \geqq 5 > e$ for $d \geqq 3$, and obtain the first part of our theorem. Note that

$$x = \frac{k}{h} > h \log h \left(1 + \frac{\log \log h}{\log h - 1}\right) \geqq 5$$

and so the method is admissible. For $d = 2$ see Remark 2 below.

REMARK 1. If we wish to obtain a result for lower x, e.g. $x = 1$, we have to use the original form of $\varphi(x)$ in (2.4), e.g., $\varphi(1) = 2\log 2 - \frac{1}{2}$. Hence for $x = 1$,

$$\frac{1}{n} \log \dim T > \log h - \varphi(1) \geqq \log(d-1)^2$$

which yields $h \geqq 4e^{-1/2}(d-1)^2$, $\approx 2.42(d-1)^2$, and proves the second part of the theorem.

REMARK 2. If R satisfies an identity of degree $d = 2$, then it evidently satisfies an identity of degree 3. But a simpler method follows by noting that then R satisfies either $S_2 = x_1x_2 - x_2x_1 = 0$ or $x_1x_2 + x_2x_1 = 0$ (or both) and hence R will always satisfy $S_2^2[x_1, x_2] = 0$.

4. A bound for $\dim D_\lambda$

Let $\lambda = (\lambda_r, \lambda_{r-1}, \cdots, \lambda_1)$, $\lambda_r \geqq \lambda_{r-1} \geqq \cdots \geqq \lambda_1 \geqq 1$ be a partition of $n = \lambda_1 + \cdots + \lambda_r$, and let D_λ be its corresponding Young diagram.

Two cases will be considered in this section. (i) The number of parts r is small relative to n; (ii) D_λ lies in a hook of width h.

To this end we use the Frobenius–Young formula for $\dim D_\lambda$:

$$\dim D_\lambda = n! \frac{\prod_{j<i} (\hat{\lambda}_j - \hat{\lambda}_i)}{\hat{\lambda}_1! \hat{\lambda}_2! \cdots ! \hat{\lambda}_r} \tag{4.1}$$

where $\hat{\lambda}_j = \lambda_j + j - 1$; we put it in an equivalent form:

$$\dim D_\lambda = \frac{n!}{\lambda_1! \cdots \lambda_r!} \frac{\prod\prod (\lambda_j - \lambda_i + j - i)}{\prod\prod (\lambda_j + j - i)} = \binom{n}{\lambda_1, \lambda_2, \cdots, \lambda_r} \cdot C \text{ and } C = C(\lambda). \tag{4.2}$$

Since

$$\frac{\hat{\lambda}_j!}{\lambda_j!}=\prod_{i=1}^{j-1}(\lambda_j+i)=\prod_{i=1}^{j-1}(\lambda_j+j-i),$$

both products of the numerator and denominator of the constant C range over $\Pi_{j=2}^{r}\Pi_{i=1}^{j-1}$; and hence $C\leqq 1$. On the other hand, since $\lambda_1\leqq\lambda_j\leqq n$,

$$C\geqq\prod\prod\frac{j-i}{\lambda_j+j-1}\geqq(n+1)^{-\rho}\tag{4.3}$$

where $\rho=\Sigma(j-i)=r(r-1)/2$. (By using the condition that $\lambda_i\geqq 1$, one can obtain that $C\leqq(n/n+1)^{\rho}$.)

Next we look for an asymptotic formula for $\binom{n}{\lambda_1,\cdots,\lambda_r}$, and to this end we use the classical integral approximation for $n!$ and $\lambda_j!$, that is,

$$\begin{aligned}\log\binom{n}{\lambda_1,\cdots,\lambda_r}&=\sum_{\nu=1}^{n}\log\nu-\sum_{i=1}^{r}\sum_{\nu=1}^{\lambda_i}\log\nu\\ &\geqq n(\log n-1)-\sum_{i=1}^{r}\lambda_i(\log\lambda_i-1)-\tfrac{1}{2}\sum\log\lambda_i\\ &\geqq\sum\lambda_i\log\frac{n}{\lambda_i}-\frac{r}{2}\log n\end{aligned}\tag{4.4}$$

since $\Sigma\lambda_i=n$.

Combining the previous inequalities we finally get

$$\frac{1}{n}\log\dim D_\lambda\geqq\sum_{i=1}^{r}\frac{\lambda_i}{n}\log\frac{n}{\lambda_i}-\frac{1}{n}\log P(n)\tag{4.5}$$

where $P(n)$ is a polynomial of n of degree $\leqq r^2$. As we shall be mainly interested in the case $n\to\infty$ and r bounded, a slightly more detailed analysis of (4.2), (4.3) and (4.4) will yield

$$\frac{1}{n}\log\dim D_\lambda=\sum_{i=1}^{r}\frac{\lambda_i}{n}\log\frac{n}{\lambda_i}+O\left(\frac{\log n}{n}\right).\tag{4.6}$$

This is the first step in proving the following theorem:

THEOREM 5. *If $\{D_\lambda\}$ ranges over a sequence of partitions $(\lambda)=(\lambda_r,\cdots,\lambda_1)$ of n of length r, and $\underline{\lim}(\lambda_1/n)=1/c$, then $c\geqq r$. If $c=r$ then* $\lim\dim D_\lambda^{1/n}=r$, *i.e., $D_\lambda\sim r^n$ asymptotically; and if $c>r$ then* $\overline{\lim}\dim D_\lambda^{1/n}=\rho<r$.

PROOF. Since $\Sigma\lambda_i=n$, and λ_1 is the minimal λ_j, it follows that $n\geqq\lambda_1 r$. Hence $\lambda_1/n\leqq 1/r$ and, therefore, $c\geqq r$.

We set $x_i = \lambda_i / n$ and consider the term $\Sigma(\lambda_i / n)\log(n/\lambda_i)$ of (4.6) as a function $F(x) = \Sigma_{i=1}^{r} x_i \log(1/x_i)$ defined in a domain $0 < b \leqq x_1 \leqq \cdots \leqq x_r \leqq 1$, $x_1 + x_2 + \cdots + x_r = a$ and in our domain $a = 1$, $b = 1/n$. Clearly $F(x)$ is defined and obtain a maximum (and minimum) in this domain.

Proposition 5'. *$F(x)$ obtains its maximal value $a \cdot \log(r/a)$, only once in the above domain, and this at the point $x_i = a/r$, $i = 1, \cdots, r$.*

Proof. Given a point $(x) = (x_1, \cdots, x_r)$, and suppose $x_i < x_j$ for some $i \neq j$, then at a point $(x') = (x'_\nu)$, $x'_i = x_i + \delta$, $x_j = x_i - \delta$ for small δ still in this domain (and with a possible change of the indices of the x'_ν) we have

$$F(x') = F(x) + \delta \log \frac{x_j}{x_i} + O(\delta^2) \tag{4.7}$$

and so for small $\delta > 0$, $F(x') > F(x) > F(x)$ and for $\delta < 0$ (and $x_i - \delta \geqq b$), $F(x') < F(x)$. This implies that the maximum is obtained only if all x_ν are equal, i.e., at the point $(a/r, \cdots, a/r)$, and there

$$F = r\frac{a}{r} \log \frac{r}{a} \,.$$

Hence,

Corollary 5''. *If $\{D_\lambda\}$ ranges over a sequence of Young diagrams of r rows with length of rows $\lambda_i = n/c_i + o(n)$, then $r \leqq c_1 \leqq \cdots \leqq c_r \leqq n$ and $\Sigma(1/c_i) = 1$,*

$$N_\lambda = \frac{1}{n} \log \dim D_\lambda = \sum \frac{\lambda_i}{n} \log \frac{n}{\lambda_i} + O\left(\frac{\log n}{n}\right) = \sum_{i=1}^{n} \frac{1}{c_i} \log c_i + o(1).$$

Next we consider diagrams D_λ which lie in a hook H of the shape given in Fig. 2, which is a hook with a middle rectangle of some size. We divide this diagram into three parts: D_1, the part of D_λ which is the horizontal leg; D_2, the part in the

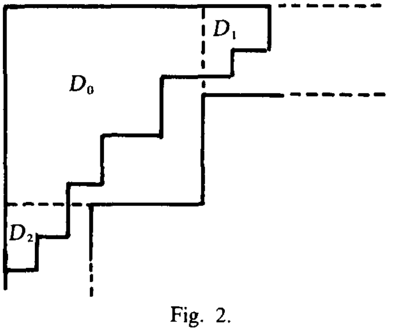

Fig. 2.

vertical leg; and D_0, within the middle rectangle. Let $|D_1| = u$, $|D_2| = v$ and $|D_0| = w$ so $n = u + v + w$. From the hook formula we obtain

$$\dim D_\lambda = \frac{n!}{\prod h_{ij}} = \frac{n!}{u!\,v!\,w!} \cdot \frac{u!}{\prod_1 h'_{ij}} \cdot \frac{v!}{\prod_2 h'_{ij}} \cdot \frac{w!}{\prod_0 h'_{ij}} \cdot \frac{\prod_1 h'_{ij} \prod_2 h'_{ij} \prod_0 h'_{ij}}{\prod h_{ij}} \tag{4.8}$$

where h'_{ij} denotes the hook number of the corresponding subdiagram D_1, D_2, D_0; Π_i is the product of the corresponding diagram D_i, and Π is the product for D_λ. Now for D_2, clearly $h'_{ij} = h_{ij}$ and we have the same hook number, and so they cancel each other in the last factor of (4.8); and similarly for the diagram D_1. For the middle part D_0, in each square we have for each quotient

$$1 \geqq \frac{h'_{ij}}{h_{ij}} \geqq \frac{1}{n}$$

since $h_{ij} \leqq n$. Hence, the last factor of (4.8) is between 1 and n^{-w}, where $w = |D_0|$.

Thus (4.8) yields

$$\begin{aligned} \binom{n}{u,\, v,\, w} \dim D_2 \dim D_1 \dim D_0 &\geqq \dim D_\lambda \\ &\geqq n^{-w} \binom{n}{u, v, w} \dim D_2 \dim D_1 \dim D_0. \end{aligned} \tag{4.9}$$

We use this inequality to prove:

THEOREM 6. *Let λ_i be the length of the rows of D_1 (horizontal part of D_λ) and μ_i the length of the columns of D_2 (vertical part of D_λ), then*

$$\frac{1}{n} \log \dim D_\lambda = \sum \frac{\lambda_i}{n} \log \frac{n}{\lambda_i} + \sum \frac{\mu_i}{n} \log \frac{n}{\mu_i} + O\left(\frac{\log n}{n}\right). \tag{4.10}$$

PROOF. From (4.6) we have

$$\frac{1}{u} \log \dim D_1 = \sum \frac{\lambda_i}{u} \log \frac{u}{\lambda_i} + O\left(\frac{\log u}{u}\right), \tag{1}$$

and if we consider the dual diagram of D_2 (i.e. the rows turned into columns) which have the same dimension, we get

$$\frac{1}{v} \log \dim D_2 = \sum \frac{\mu_j}{v} \log \frac{v}{\mu_j} + O\left(\frac{\log v}{v}\right). \tag{2}$$

Finally $\dim D_0$ is bounded by $w!$ as a diagram corresponding to the symmetric group S_w.

For the binomial factor $\binom{n}{u,v,w}$ of (4.9) we use the asymptotic formula of the factorial, namely

$$\log\binom{n}{u,\,v,\,w} = \log n! - \log u! - \log v! - \log w!$$

$$= n(\log n - 1) - u(\log u - 1) - v(\log v - 1) - w(\log w - 1) + O(\log n)$$

since u, v and $w < n$. Thus using $n = u + v + w$ we get

$$= u\log\frac{n}{u} + v\log\frac{n}{v} + O(\log n)$$

noting that in the shape H, w is a bounded number, even if we vary D_λ so that $n \to \infty$. This proves that

$$\text{(0)} \qquad \frac{1}{n}\log\binom{n}{u,\,v,\,w} = \frac{u}{n}\log\frac{n}{u} + \frac{v}{n}\log\frac{n}{v} + O\left(\frac{\log n}{n}\right).$$

If we put the values of (1), (2), (0) in (4.9) we get

$$\frac{1}{n}\log\dim D_\lambda = \frac{u}{n}\log\frac{n}{u} + \frac{u}{n}\cdot\frac{1}{u}\log\dim D_1 + \frac{v}{n}\log\frac{n}{v} + \frac{v}{n}\frac{1}{v}\log\dim D_2 + O\left(\frac{\log n}{n}\right)$$

$$= \frac{u}{n}\left(\log\frac{n}{u} + \sum\frac{\lambda_i}{u}\log\frac{u}{\lambda_i}\right) + \frac{v}{n}\left(\log\frac{n}{v} + \sum\frac{\mu_j}{v}\log\frac{v}{\mu_j}\right) + O\left(\frac{\log n}{n}\right)$$

since

$$\frac{u}{n}O\left(\frac{\log u}{u}\right) = O\left(\frac{\log u}{n}\right) = O\left(\frac{\log n}{n}\right) \quad \text{etc.}$$

Now $\Sigma\,\lambda_i = u$, $\Sigma\,\mu_j = v$. Hence we have

$$\frac{1}{n}\log\dim D_\lambda = \frac{u}{n}\sum\frac{\lambda_i}{u}\left(\log\frac{u}{\lambda_i} + \log\frac{n}{u}\right) + \frac{v}{n}\sum\frac{\mu_j}{v}\left(\log\frac{v}{\mu_j} + \log\frac{n}{v}\right) + O\left(\frac{\log n}{u}\right)$$

$$= \sum\frac{\lambda_i}{n}\log\frac{n}{\lambda_i} + \sum\frac{\mu_j}{n}\log\frac{n}{\mu_j} + O\left(\frac{\log n}{n}\right)$$

which proves (4.10).

COROLLARY 6′. *If h, k are the respective width of the horizontal and vertical legs of H* (Fig. 2), *then*

$$\dim D_\lambda^{1/n} \leqq (h+k)\left(1 + O\left(\frac{\log n}{n}\right)\right),$$

and if $v = |D_2| = o(n)$, the number of squares in the vertical leg, then $\dim D_\lambda^{1/n} \leqq h(1+o(1))$.

PROOF. Let h_0, k_0 be the respective number of rows and column of D_λ, then it follows from (4.10), from the fact that $\Sigma\lambda_i = u$, $\Sigma\mu_j = v$, and by Proposition 5′, that

$$\frac{1}{n}\log\dim D_\lambda \leqq \frac{u}{n}\log\left(h_0\frac{n}{u}\right)+\frac{v}{n}\log\left(k_0\frac{n}{v}\right)+O\left(\frac{\log n}{n}\right). \tag{4.11}$$

Since $\log x$ is a convex function, we have

$$\frac{a}{a+b}\log x+\frac{b}{a+b}\log y \leqq \log\frac{ax+by}{a+b}$$

for a, b and x, y positive. In our case we get for $a = u/n$, $b = v/n$, and from the fact $u+v+O(1)=n$, that

$$\frac{1}{n}\log\dim D_\lambda \leqq \frac{u+v}{n}\log\frac{(h_0+k_0)}{u+v}n \leqq \log(h_0+k_0)+O\left(\frac{1}{n}\right)$$

which proves the first part of Corollary 6′, since $h_0 \leqq h$, $k_0 \leqq k$.

The second part follows since $v = o(n)$, $u = n+o(n)$, and as $x\log x \to 0$ when $x \to 0$ the second factor of (4.11) is $o(1)$.

5. The codimension series

Given the codimension series $\{c_n(R)\}$ of a PI-ring R, we consider $c = \underline{\lim}\, c_n(R)^{1/n}$. It seems probable that the $\{c_n(R)^{1/n}\}$ has a limit, and in one case, Regev [5] has recently proved that for the matrix ring $M_r(K)=R$ then $\lim c_n(R)^{1/n} = r^2$. It follows also from [3] that this limit exists for the exterior algebra. For general rings we can only show that $c = \underline{\lim}\, c_n(R)^{1/n}$ can replace the Latyshef bound $(d-1)^2$ (which is $\geqq c$) in the main theorem of [4]; namely,

THEOREM 7. (1) *For every integer $h > c$, there exists k such that R satisfies the identity $S_h[x_1,\cdots,x_n]^k = 0$ (and $S_k^h = 0$).*

(2) *If $c > 0$, then for every integer $h > c\log c$ and if $c = 0$ for any $h \geqq 1$, the ring $R/N_1(R)$ satisfies the identity $S_h[x_1,x_2,\cdots,x_h]=0$; and the principal ideal generated by any finite number of values $S_h[a_{ji},\cdots,a_{jh}]$, $i=1,\cdots,t$ is nilpotent of index depending only on t and h.*

(3) *If R has either no right or left annihilator (e.g. $1\in R$), then for every integer $h > c^2$, there exists an integer k such that R satisfies all identities I_D corresponding to a Young diagram which contains either the rectangle $T=(k^h)$ or $T'=(h^k)$.*

PROOF. The proof of the three parts will follow the idea of section 2 by showing that

$$\frac{1}{n}\log\dim D > \frac{1}{n}\log c_n(R)$$

for certain n's and appropriate diagrams D, so that $I(R)\supseteq I_D$ and then we apply (3.2)-(3.4).

Let $c_0 > c$ be a number (to be fixed later) and let $\{n_\lambda\}$ be a sequence of integers such that $c_n(R)^{1/n}\leqq c_0$, which exist by definition of $c=\underline{\lim}\, c_n(R)^{1/n}$.

Given $n\in\{n_\lambda\}$ we choose $k=[n/h]$, i.e. $k\leqq n<h(k+1)$. Let D be any Young diagram of content n, which contains the rectangle $T=(k^h)$. Hence, $\dim D\geqq\dim T$ (e.g. [1]). We shall choose c_0 so that $\dim D\geqq\dim T\geqq c_0^n> c_n(R)$, where $n=|D|<(k+1)h$, and then apply (3.2).

Indeed, by Theorem 1

$$\frac{1}{n}\log\dim D\geqq\frac{1}{n}\log\dim T=\frac{kh}{n}\left(\frac{1}{kh}\log\dim T\right)$$

$$>\frac{kh}{n}(\log h-\varphi(x))\geqq\frac{k}{k+1}(\log h-\varphi(x))$$

as $n\to\infty$, also $x\to\infty$ and so $k/(k+1)\to 1$ and $\varphi(x)\to 0$. Hence the last term tends to $\log h$. Consequently, if we choose any c_0 so that $h>c_0>c$ we can find a large $n\in\{n_\lambda\}$ and $x=k/h$ so that

$$\frac{1}{n}\log\dim T\geqq\frac{k}{k+1}(\log h-\varphi(x))\geqq\log c_0>\log c_n(R)^{1/n}.$$

Next we apply (3.3) and obtain that R will satisfy the identity $S_h^k[x_1,\cdots,x_j]x_{kh+1}\cdots x_n=0$, from which it follows that R satisfies the identity $S_h[x]^{k+1}=0$ which proves (1).

To prove (2), we follow the same idea: for a given c_0 we shall choose $n\in\{n_\lambda\}$ so that $c_n(R)<c_0^n$.

But now we choose $k=[n/(h+1)]$, i.e. $k(h+1)\leqq n<(k+1)(h+1)$, then for any diagram $D\geqq T=(k^h)$ of content n, we have as before

$$\frac{1}{n}\log\dim D\geqq\frac{kh}{n}\left(\frac{1}{kh}\log\dim T\right)\geqq\frac{kh}{(k+1)(h+1)}(\log h-\varphi(x)).$$

If $n\to\infty$ (in the sequence $\{n_\lambda\}$), then also $x=k/h\to\infty$ and the last term of the inequality tends to $(h/(h+1))\log h$.

We need the fact that if $h>c+\log c$ then

$$\frac{h}{h+1}\log h > \log c,$$

and indeed for

$$\frac{h}{h+1}\log h - \log c = \frac{1}{h+1}[h(\log h - \log c) - \log c] \geqq \frac{1}{h+1}[h - c - \log c] \geqq 0,$$

since

$$h(\log h - \log c) = h\int_c^h \frac{dt}{t} \geqq h - c \qquad \text{for } h > c.$$

If $h > c + \log c$ we can find $c_0 > c$ and $h > c_0 + \log c_0$. Thus, it follows that

$$\frac{h}{h+1}\log h > \log c_0 > \log c.$$

Consequently, as before we can find a sequence $\{n_\lambda\}$ and for $n \in \{n_\lambda\}$, we get the corresponding k, x so that (for appropriate $\varepsilon > 0$)

$$\frac{1}{n}\log \dim D \geqq \frac{h}{h+1}\log h - \varepsilon \geqq \log c_0 \geqq \log c_n(R)^{1/n}.$$

Hence by Proposition 3 it follows that R satisfies

$$S_h[x]y_1S_h[x]y_2\cdots S_h[x]y_k = 0$$

which proves that every principal ideal in R generated by an element $S_h[a_1,\cdots,a_h]$ is nilpotent of exponent $\leqq k$. From this, we easily conclude that also any ideal generated by t elements is nilpotent of exponent $\leqq k\cdot t$.

To prove the last part we need a simple lemma:

LEMMA 8. *If R has no right or has no left annihilator, then the sequence $\{c_n(R)\}$ is non-decreasing.*

Indeed, let R have no right annihilator, then if $f[x_1,\cdots,x_n]x_{n+1} = 0$ in R also $f[x_1,\cdots,x_n] = 0$ is an identity of R. Thus, the mapping $f \to fx_{n+1}$ induces an injection of $V_n(R)/I_n(R)$ into $V_{n+1}(R)/I_{n+1}(R)$, which proves that $c_n(R) \leqq c_{n+1}(R)$.

To prove part (3), we choose c_0 so that $h > c_0^2 > c^2$ and $k = [n/2h]$ for $n \in \{n_\lambda\}$, i.e., $2kh \leqq n < 2(k+1)h$, or equivalently

$$\frac{1}{2} \geqq \frac{kh}{n} > \frac{1}{2}\frac{k}{k+1}.$$

As before, let $D \geqq T = (k^h)$ and $|D| = m$, where $n \geqq 2kh > m \geqq kh$. Then first we have

$$\frac{n}{m} \log c_0 > \frac{n}{m}\left(\frac{1}{n} \log c_n(R)\right) = \frac{1}{m} \log c_n(R) \geqq \frac{1}{m} \log c_m(R).$$

On the other hand,

$$\frac{1}{m} \log \dim D \geqq \frac{kh}{m}\left(\frac{1}{kh} \log \dim T\right) \geqq \frac{n}{m} \cdot \frac{kh}{n}(\log h - \varphi(x))$$

$$\geqq \frac{n}{m} \cdot \frac{k}{2(k+1)}(\log h - \varphi(x)).$$

In order to achieve $\dim D \geqq c_m(R)$, we choose $n \in \{n_\lambda\}$ and, respectively, k and x, so that

$$\frac{k}{2(k+1)}(\log h - \varphi(x)) \geqq \log c_0,$$

which is possible since the left side tends to $\geqq \frac{1}{2} \log h$ which is $\geqq \log c_0$. The rest of the proof follows by (3.4). The second half of part (3) is a consequence of the observation that $\dim(k^h) = \dim(h^k)$, and all computations are therefore symmetrical.

6. Width of the hook

Part (3) of the preceding theorem shows that given any integer $m > c^2$, e.g. $m = [c^2] + 1$, then there exists an integer k such that for all diagrams $D \geqq (k^m)$ or $D \geqq (m^k)$, the identity $I(R)$ contains I_D. Let $\chi_n(R)$ be the co-character of R, i.e., the character of the representation module $V_n / I_n(R)$, and let

$$\chi_n(R) = \sum a_D \chi_D \qquad \text{where } a_D \neq 0, \tag{6.1}$$

then the preceding result means that these diagrams D satisfy $D \not\geqq (k^m)$ and $D \not\geqq (m^k)$. In other words, we can state this fact:

COROLLARY 9. *The diagrams of D of the co-characters of* (6.1) *lie in a hook H of the shape in Fig.* 3, *and the width of its legs is $p = [c^2]$ where $c = \lim c_n(R)^{1/n}$.*

Note that the size of the maximal square in H cannot be determined by our method.

We quote an important result of Berele and Regev [2], stating that the coefficients of the co-characters $\{a_D\}$ of (6.1) satisfy $\Sigma a_D = O(n^s)$ for some

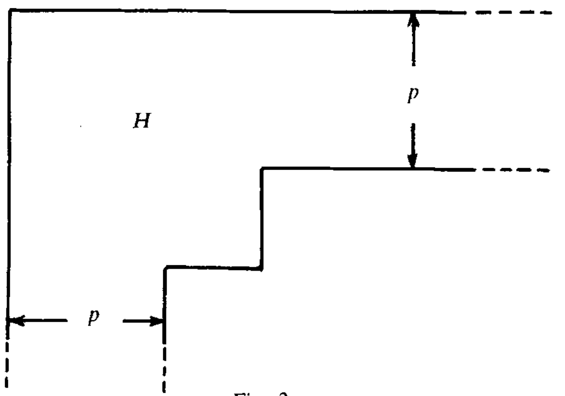

Fig. 3.

integer s, i.e. Σa_D has polynomial growth. Hence it follows from (6.1) that $c_n(R) = \Sigma a_D \dim D$ and, therefore,

$$\max\dim D \leqq c_n(R) \leqq \max\dim D \cdot \sum a_D \leqq O(n^s) \cdot \max\dim D$$

which implies

$$\max\dim D^{1/n} \leqq c_n(R)^{1/n} \leqq \max\dim D^{1/n}\left(1 + O\left(\frac{\log n}{n}\right)\right). \tag{6.2}$$

Since these diagrams lie within the hook H whose width is $[c^2]$, it follows by Theorem 6 that

$$\dim D^{1/n} \leqq 2[c^2] + O\left(\frac{\log n}{n}\right),$$

which proves one part of the following result:

COROLLARY 10. *If R has no right or left annihilator, then*

$$c = \underline{\lim}\, c_n(R)^{1/n} \leqq \overline{\lim}\, c_n(R)^{1/n} \leqq 2[\underline{\lim}\, c_n(R)^{1/n}]^2;$$

and if R satisfies the Cappelli identity, then

$$\overline{\lim}\, c_n(R)^{1/n} \leqq [c^2].$$

Indeed, the first part follows by passing to $\overline{\lim}$ on both sides of (6.2). The second part of the theorem follows from the fact that in this case H can be replaced by a strip and then

$$\dim D^{1/n} \leqq [c^2] + O\left(\frac{\log n}{n}\right),$$

and the rest is as in the first part.

An upper bound for $C = \overline{\lim}\, c_n(R)^{1/n}$ can be given by Latyshef's result to be $(d-1)^2$, since $c_n(R) \leqq (d-1)^{2n}$, where d is the minimal degree of the identities of R. A lower bound for $\underline{\lim}\, c_n(R)^{1/n}$ can be obtain by a recent result of Regev [5]:

For the matrix ring $R = M_r(K)$, $\lim c_n(R)^{1/n} = r^2$, i.e., for these rings, there is always a limit to $c_n(R)^{1/n}$. Using this result we can prove:

THEOREM 10. *For a PI-algebra R let $N_1(R)$ be the sum of nilpotent ideals of R, then $R/N_1(R)$ can be embedded in some matrix ring $M_s(K)$, K commutative, semi-prime, and $c = \lim c_n(R)^{1/n} \geqq s^2$.*

PROOF. Let $\bar{R} = R^R$ be the product ring, i.e., the ring of all functions from R to R, and let $L(\bar{R})$ be its lower radical. Consider the sequence of maps $\psi : R \to \bar{R} \to \bar{R}/L(\bar{R})$, where ψ is the diagonal embedding of R into $\bar{R}$ follows by the canonical projection. We prove Ker $\psi = N_1(R)$: indeed, if $a \in \operatorname{Ker}\psi$ then for the function $f \in \bar{R}$, defined by $f(r) = r$ for every $r \in R$, we have that $\bar{a}f \in L(\bar{R})$, where $\bar{a} \in R^R$ is the image of $a \in R$, i.e. $\bar{a}(r) = a$ for all $r \in R$. $L(\bar{R})$ is nil and so $\bar{a}f$ is nilpotent, i.e., there exists m such that $(\bar{a}f)^m = 0$, hence for all $r \in R$, $(\bar{a}f)^m(r) = (ar)^m = 0$. This yields that aR is a nil ideal of bounded index, but R is an algebra of characteristic zero, hence by the Nagata–Higman theorem it is nilpotent, which implies that $a \in N_1(R)$, i.e., $\operatorname{Ker}\psi \subseteq N_1(R)$.

Conversely, if $a \in N_1(R)$, then a generates a nilpotent ideal aR of index m, then $\bar{a}\bar{R}$ is also nilpotent, since

$$(\bar{a}f_1 \cdots \bar{a}f_m)(r) = af_1(r)af_2(r) \cdots af_m(R) \in (aR)^m = 0.$$

Thus $a \in \operatorname{Ker}\psi$, and so $N_1(R) \subseteq \operatorname{Ker}\psi$.

Now $\bar{R}/L(\bar{R})$ is a semi-prime PI-ring, and as such it is embeddable in some matrix ring $M_s(K)$, K commutative and semi-prime and which satisfies the same identities. Hence $c_n(\bar{R}/L(\bar{R})) = c_n(M_s(K)) \cong s^{2n}$. On the other hand, the identities of $\bar{R}$ and R are the same so that $c_n(R) = c_n(\bar{R})$, and clearly $c_n(\bar{R}) \geqq c_n(\bar{R}/L(\bar{R})) \cong s^{2n}$, so that $\underline{\lim}\, c_n(R)^{1/n} \geqq s^2$. q.e.d.

This number s of Theorem 10 can also be characterised in a different way.

THEOREM 11. *The integer s of Theorem 10 is the maximal order of matrix ring $M_s(Q)$ which satisfies all identities of R, i.e., $I(M_s(Q)) \supseteq I(R)$.*

PROOF. We can replace R by the universal ring $F\langle x\rangle/I(R)$, where $F\langle x\rangle$ is the free ring in an infinite number of indeterminates, since the identities of R and of the universal ring coincide. For the universal ring we have $N_1(R) = I(M_s(Q))/I(R)$. Indeed, $N_1(R)$ is the lower radical $L(R)$ of R, since $L(R) \supseteq N_1(R)$ and if $f(x) \in L(R)$, then $(f[x]x_{n+1})^m = 0$ in R for some m and x_{n+1} which is not in the indeterminates of $f[x]$. In other words this means that $f[x]R$ is a nil ideal of bounded index, and since the characteristic is zero, $f[x] \in N_1(R)$. Now, for the universal ring $L(R)$ is the identity of the matrix ring, i.e. $L(R) = I(M_t(Q))/I(R)$, and t is the maximal integer for which $I(M_t(Q)) \geqq I(R)$. One can now easily verify that $t = s$, the integer of Theorem 10.

Corollary 12. *If* $\lim c_n(R)^{1/n} < 4$, *then* $R/N_1(R)$ *is commutative, and if* $R/N_1(R)$ *is not commutative, then* $\lim c_n(R)^{1/n} \geqq 4$.

Indeed, the first part follows since $s^2 < 4$ implies $s = 1$. The second part follows since then $R/L(R)$ satisfies the same identities of a matrix ring $M_t(K)$, $t \geqq 2$, and so $c_n(R)^{1/n} \geqq c_n(R/L(R))^{1/n} \to t^2$. Hence $\underline{\lim}\, c_n(R)^{1/n} \geqq t^2 \geqq 4$.

7. Lim sup of the sequence $\{c_n(R)^{1/n}\}$

Let $C = \overline{\lim}\, c_n(R)^{1/n}$; we determine a lower bound for the ultimate width of the hook in which the diagrams D of the co-characters $\chi_n(R)$ of (6.1) lie in terms of C if $C > 0$. More precisely:

Theorem 13. *For any integer* N, *there exist diagrams* D *of the co-character such that* D *contains either a hook* T *of width* k *and* h, *or a strip* $T = (a^h)$ *or* (h^a) *such that* $|T| > N$ *and* $h + k \geqq C$, *and* $h \geqq C$ *for the case of a strip* T (Fig. 4).

We need a few preliminary results.

Lemma 13'. *Let* D *be a Young diagram of content* n, *divided into a union of diagrams* D_i, $|D_i| = d_i$, $i = 0, 1, 2, \cdots, r$; *then*

$$\dim D \leqq \binom{n}{d_0, d_1, \cdots, d_r} \dim D_0 \dim D_1 \cdots \dim D_r.$$

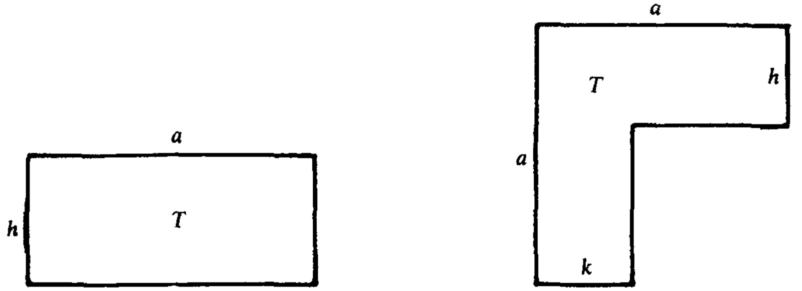

Fig. 4.

The proof follows the same method as that of (4.9):

$$\dim D = \frac{n!}{\Pi h_{ij}} = \frac{n!}{d_0! \cdots d_r!} \Pi_{(\rho)} \frac{d_\rho!}{h_{ij}^{(\rho)}} \prod_\rho \Pi_{(\rho)} \frac{h_{ij}^{(\rho)}}{h_{ij}}$$

where $\Pi_{(\rho)}$ is the product ranging over the subdiagram D_ρ, with hook numbers $h_{ij}^{(\rho)}$. Clearly $h_{ij}^{(\rho)} \leqq h_{ij}$, so that the last factor is $\leqq 1$, which proves the lemma.

We are going to use this lemma in a special case where all D_i, except D_0, are either rows or columns of length d_i and width 1 (Fig. 5). In this case we show:

COROLLARY 13″. $\dim D \leqq (n/(h+k))^{d_0}(h+k)^n$.

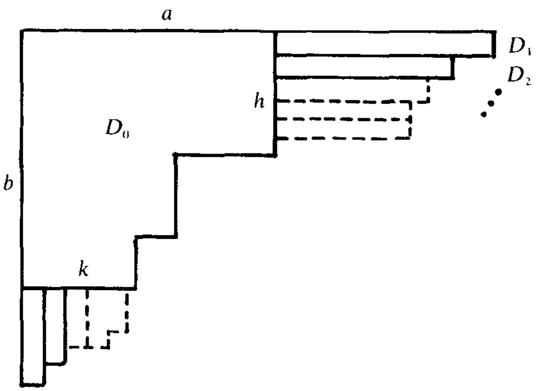

Fig. 5.

Indeed, by the preceding lemma we have, in our case, $\dim D_i = 1$ for $i > 1$. Also $\dim D_0 \leqq d_0!$ since $|D_0| = d_0$. Hence it follows by Lemma 13′ that

$$\dim D \leqq \binom{n}{d_0, d_1, \cdots, d_{h+k}} d_0! = \binom{n}{d_0} d_0! \binom{n-d_0}{d_1, d_2, \cdots, d_r} \leqq n^{d_0}(h+k)^{n-d_0}.$$

We turn to the proof of the theorem. By the statement preceding (6.2) it follows that there exist constants K and s so that $c_n(R) < Kn^s \dim D$ for some diagram D which lies in the co-character $\chi_n(R)$. Next, the definition of C implies that given $\varepsilon > 0$ there exists an infinite sequence $\{n_\lambda\}$ such that for $n \in \{n_\lambda\}$, $c_n(R) > (C - \varepsilon/2)^n$.

Using the last corollary we obtain

$$(7.1) \qquad (C-\varepsilon/2)^n \leqq c_n(R) \leqq Kn^s n^{d_0}(h+k)^n.$$

The procedure of choosing the diagrams D is as follows:

Given N and $\varepsilon > 0$, find a sequence $\{n_\lambda\}$ such that for all $n \in \{n_\lambda\}$, $c_n(R) \geqq (C - \varepsilon/2)^n$. For each $n \in \{n_\lambda\}$ choose a diagram D such that $c_n(R) \leqq Kn^s \dim D$. Now we consider two cases.

Case 1. These diagrams D have unbounded height and width. In this case we choose integer $a = b$ in Fig. 5, which will be chosen later depending on N and C, and R. To each D we cut the diagram at height and width a, and set D_0 to be the remaining squares; then $d_0 = |D_0| \leqq a^2$. Next we choose $n \in \{n_\lambda\}$ large enough such that

$$h + k > (C - \varepsilon/2)(Kn^{s+a^2})^{-1/n} \geqq C - \varepsilon$$

which is possible for large n in the sequence $\{n_\lambda\}$, since by the lemma

$$(C - \varepsilon/2)^n \leqq Kn^{s+d_0}(h+k)^n \leqq Kn^{s+a^2}(h+k)^n.$$

If we choose $\varepsilon > 0$ small enough so that $C - \varepsilon > [C]$ if $C \neq [C]$, we get $h + k \geqq C$ since $h + k$ is an integer; and if C is an integer and $C - \varepsilon > C - 1$, we also have $h + k \geqq C$.

Finally, if T is the hook in D based on k and h we get

$$|T| = (k+h)a - kh \geqq (k+h)a - p^2$$

where p is an upper bound for the hook of shape H (Fig. 3) in which all these diagrams D lie. Hence, if we choose $a > (N + p^2)/C$ we have $|T| > N$ as required.

Case 2. The chosen diagrams have either a bounded height or width (say of bounded height). Then we choose b in Fig. 5 to be the maximum height. We repeat the preceding procedure with $k = 0$, and take the leg of D cut at distance a (Fig. 5); then $d_0 = |D| \leqq ab$. Also, choose n large enough so that

$$h > (C - \varepsilon/2)(Kn^s n^{ab})^{-1/n} \geqq C - \varepsilon,$$

hence, as before, $h \geqq C$.

Finally, in this case we obtain the strip T of height h and length a, so that $|T| \geqq ha > N$ if we choose $a > N/C$.

8. Appendix 1. An asymptotic formula

The lower bound of $\dim(k^h)$ given in section 1 is not far from the asymptotic formula, which proves that we cannot expect that this computational method of finding h, k, so that $S_h^k[x] = 0$ holds, will yield any result for $h < 1 + (d-1)^2$, although we know it holds for $h = [d/2]$.

THEOREM 14. $\dim[k^h] = C(k+h)^{1/12} n^{5/12} (e^{-\varphi(x)} h)^n (1 + O(1/n))$; $x = k/h$, $n = kh$, *where* $\varphi(x)$ *is the function in* (2.4), *and some constant* $C = C_0(1 + O(1/h))$.

We use the Stirling formula

$$\sum_{\nu=1}^{n} \log \nu = n(\log n - 1) + \tfrac{1}{2} \log n + O(1). \tag{8.1}$$

Following the formulas of (2.2) we have

$$\begin{aligned}\sum \log(s+t-1) &= \sum_{t=1}^{k} \sum_{s=1}^{h} \log(s+t-1)\\ &= \sum_{\nu=1}^{k+h} (k+h-\nu)\log \nu - \sum_{\nu=1}^{k} (k-\nu)\log \nu - \sum_{\nu=1}^{h} (l-\nu)\log \nu\\ &= g(k+h) - g(k) - g(h)\end{aligned}$$

where $g(m) = \sum_{\nu=1}^{m} (\mu - \nu)\log \nu$. The last formula is obtained by setting $s + t - 1 = \nu$ and summing in three areas, $\nu < k$ (say $k \geqq h$), $k \geqq \nu > h$, and $\nu \leqq h$. We then get

$$\begin{aligned}\sum_{t=1}^{k} \sum_{s=1}^{h} &= \sum_{\nu=k+1}^{k+h-1} (k+h-\nu)\log \nu + \sum_{\nu=h+1}^{k} h \log \nu + \sum_{\nu=1}^{h} \nu \log \nu\\ &= \sum_{\nu=1}^{k+h} (k+h-\nu)\log \nu + \sum_{\nu=h+1}^{k} \log \nu + \sum_{\nu=1}^{h} \nu \log \nu\end{aligned}$$

and the rest is immediate (when $h = k$ the empty sum is taken to be zero).

Next we obtain an asymptotic formula for $g(m)$. First, we need the well known formula

$$\begin{aligned} m \sum_{\nu=1}^{m} \log \nu &= m[m(\log m - 1) + \tfrac{1}{2} \log m + C + O(1/m)]\\ &= m^2 \log m - m^2 + \tfrac{1}{2} m \log m + Cm + O(1).\end{aligned}$$

Then

$$\begin{aligned}\sum_{\nu=1}^{m} \nu \log \nu &= \tfrac{1}{2} \sum_{\nu=1}^{m} [\nu^2 - (\nu-1)^2 + 1]\log \nu\\ &= \tfrac{1}{2}\left[\sum_{\nu=1}^{m} \nu^2 \log \nu - \sum_{\nu=1}^{m-1} \nu^2 \log(\nu+1) - \sum_{\nu=1}^{m} \log \nu\right]\\ &= \frac{m^2}{2} \log m - \tfrac{1}{2} \sum_{\nu=1}^{m-1} \nu^2 \log \frac{\nu+1}{\nu} + \tfrac{1}{2} \sum_{\nu=1}^{m} \log \nu.\end{aligned}$$

Note that

$$\log\frac{\nu+1}{\nu}=\frac{1}{\nu}-\frac{1}{2\nu^2}+\frac{1}{3\nu^3}-\frac{\theta(\nu)}{\nu^4}, \qquad 0\leqq\theta<1.$$

Hence

$$\begin{aligned}\sum_{\nu=1}^{m}\nu\log\nu&=\frac{m^2}{2}\log m-\frac{m(m-1)}{4}+\frac{m-1}{4}-\tfrac{1}{6}[\log m+C+O(1/m)]+O(1)\\&\quad+\tfrac{1}{2}[m(\log m-1)+\tfrac{1}{2}\log m+O(1)]\\&=\frac{m^2}{2}\log m-\frac{m^2}{4}+\frac{m}{2}+\tfrac{1}{2}m\log m+\tfrac{1}{12}\log m+O(1)\end{aligned}$$

where C is Euler's constant. Hence

$$g(m)=\sum_{\nu=1}^{m}(m-\nu)\log\nu=\frac{m^2}{2}\log m-\tfrac{3}{4}m^2-\tfrac{1}{12}\log m+Am+O(1)$$

for some constant A.

Finally, by (8.1) for $n=kh$, $x=k/h$, we obtain that

$$\begin{aligned}\log\dim(k^h)&=[n(\log n-1)+\tfrac{1}{2}\log n+O(1)]\\&\quad-[\tfrac{1}{2}(k+h)^2\log(k+h)-\tfrac{3}{4}(k+h)^2-\tfrac{1}{12}\log(k+h)-A(k+h)]\\&\quad+[\tfrac{1}{2}k^2\log k-\tfrac{3}{4}k^2-\tfrac{1}{12}\log k+Ak]\\&\quad+[\tfrac{1}{2}h^2\log h-\tfrac{3}{4}h^2-\tfrac{1}{12}\log h+Ah]+O(1)\\&=n[\log h-\varphi(x)]+\tfrac{1}{2}\log n+\tfrac{1}{12}\log\left(\frac{k+h}{kh}\right)+O(1)\end{aligned}$$

which proves that, for $n=kh$,

$$\dim(k^h)=(e^{-\varphi(x)}h)^n n^{5/12}(k+h)^{1/12}g(n,h).$$

If we use one additional term in the approximation of $\Sigma\log\nu$ and $\Sigma\,\nu\log\nu$, we can show that $g(n,h)=C(1+O(1/n))$ for some constant $C=C(h)$, which is of the form $C_0(1+O(1/h))$.

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